MMAT5360 200420

(A, v) : $A = \{1, 2, 3, \cdots, n\}$

$v : \mathcal{P}(A) \to \mathbb{R}$ characteristic function

1. $v(\phi) = 0$

2. Superaddivity: If $S \cap T = \phi$

$$v(S \cup T) \geq v(S) + v(T)$$

Shapley value

$$\phi_k = \sum_{S \in \mathcal{P}(A) \setminus \{\phi\}} \frac{(n-s)!(s-1)!}{n!} \delta(k, S)$$

$$\delta(k, S) = v(S) - v(S \setminus \{k\}), \quad s = |S|$$

ϕ_k measures the average contribution over all permutations in which the grand coalition is formed

$n=3$: If $v(\{1\})=v(\{2\})=v(\{3\})=0$,

$$\phi_1 = \frac{2v(\{1,2,3\}) + v(\{1,2\}) + v(\{1,3\}) - 2v(\{2,3\})}{6}$$

Thm: $$\phi_k = \frac{1}{n!}\sum_{\sigma\in S_n}\left(v(S_k^\sigma) - v(S_k^\sigma \setminus \{k\})\right)$$

S_n = set of all permutations of $1,2,3,\dots,n$

$|S_n| = n!$

$$\sigma(1),\sigma(2),\dots,\sigma(i-1),\underbrace{\sigma(i)}_{= k},\sigma(i+1),\dots,\sigma(n)$$

$$S_k^\sigma = \{\sigma(1),\sigma(2),\dots,\sigma(i)=k\}$$

Example 3-person constant sum game

$n = 3$

Coalitions containing 1:

$\{1\}, \{1,2\}, \{1,3\}, \{1,2,3\}$

$$\phi_1 = \frac{(3-1)!(1-1)!}{3!}\left(v(\{1\}) - v(\emptyset)\right) + \frac{(3-2)!(2-1)!}{6}\left(v(\{1,2\}) - v(\{2\})\right)$$

$$+ \frac{(3-2)!(2-1)!}{3!}\left(v(\{1,3\}) - v(\{3\})\right) + \frac{(3-3)!(3-1)!}{3!}\left(v(\{1,2,3\}) - v(\{2,3\})\right)$$

$$= \frac{2}{6}\left(v(\{1\}) - 0\right) + \frac{1}{6}\left(v(\{1,2\}) - v(\{2\})\right)$$

$$+ \frac{1}{6}\left(v(\{1,3\}) - v(\{3\})\right) + \frac{2}{6}\left(v(\{1,2,3\} - v(\{2,3\})\right)$$

Example (Used car game)

$v(\{1\}) = v(\{2\}) = v(\{3\}) = 0$

① $\phi_1 = \frac{2}{6} \times 0 + \frac{1}{6}(500-0) + \frac{1}{6}(700-0) + \frac{2}{6}(700-0)$

$$= \frac{1300}{3} \simeq 433.33$$

Similarly: $\phi_2 = \frac{250}{3} \simeq 83.33$, $\phi_3 = \frac{550}{3} \simeq 183.33$

② $\phi_1 = \frac{2 \times 700 + 500 + 700 - 2 \times 0}{6} = \frac{1300}{3}$

Example (Voting game)

Shapley-Shubik index

$A = \{R, B, G, W\}$

No. of votes:

R	B	G	W
40	30	25	5

$v(S) = 1$ winning coalition

$v(S) = 1$ winning coalition

$= 0$ otherwise

Winning coalitions: $\{R, B\}$, $\{R, G\}$, $\{B, G\}$,

any 3 of them, 4

$$\phi_R = 2 \times \frac{(4-2)!(2-1)!}{4!}(1-0) + 2 \times \frac{(4-3)!(3-1)!}{4!}(1-0)$$

$$= \frac{1}{3}$$

$\delta(R, S) = 1$ if $S = \{R, B\}, \{R, G\}, \cancel{\{R, B, G\}}$ $\quad s = 2$

$\{R, B, W\}, \{R, G, W\}$ $\quad s = 3$

$\phi_B = \phi_G = \frac{1}{3}$, $\quad \phi_W = 0$

Example (Taxi hiring)

A, B, C want to travel from CUHK

to City One, Tai Wai, Tsuen Wan

Destination	Cost	$v(S)$ = amount saved
A	50	0
B	60	0
C	120	0
A, B	80	$50+60-80=30$
A, C	150	20
B, C	130	50
A, B, C	160	$50+60+120-160=70$

$$\phi_A = \frac{2\times 70 + 30 + 20 - 2\times 50}{6} = 15$$

$\phi_B = 30$, $\phi_C = 25$

A, B, C should pay

	Cost	Shapley value	Need to pay
A	50	15	\$35
B	60	30	\$30
C	120	25	\$95

Thm (Axioms for Shapley values)

The Shapley values $\vec{\phi}=(\phi_1, \cdots, \phi_n)$ is the unique payoff allocation which satisfies the following properties.

1. (Efficiency) $\sum_{i=1}^{n} \phi_i = v(\cancel{\phi})$

2. (Symmetry) If $i, j \in A$ s.t.

$$v(S \cup \{i\}) = v(S \cup \{j\})$$

for any S not containing i, j, then

$$\phi_i = \phi_j$$

3. (Dummy player) If $v(S \cup \{i\}) = v(S)$ for any S, then $\phi_i = 0$.

4. (Additivity) Suppose v and μ are two characteristic functions. Then

$$\vec{\phi}(v+\mu) = \vec{\phi}(v) + \vec{\phi}(\mu)$$

$$(v+\mu)(S) = v(S) + \mu(S)$$